

# Agglomerative Percolation in Two Dimensions

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PACS 64.60.ah – Percolation

PACS 68.43.Jk – Diffusion of adsorbates, kinetics of coarsening and aggregation

PACS 89.75.Da – Systems obeying scaling laws

**Abstract** – We study a process termed *agglomerative percolation* (AP) in two dimensions. Instead of adding sites or bonds at random, in AP randomly chosen clusters are linked to all their neighbors. As a result the growth process involves a diverging length scale near a critical point. Picking target clusters with probability proportional to their mass leads to a runaway compact cluster. Choosing all clusters equally leads to a continuous transition in a new universality class for the square lattice, while the transition on the triangular lattice has the same critical exponents as ordinary percolation – violating blatantly the basic notion of universality.

Percolation is a pervasive concept in statistical physics and an important branch of mathematics [1]. It typifies the emergence of long range connectivity in many systems such as the flow of liquids through porous media [2], transport in disordered media [3], spread of disease in populations [4], resilience of networks to attack [5], formation of gels [6] and even of social groups [7]. It also underlies a number of other critical phenomena – like the Ising order/disorder transition, which is a percolation transition on the set of spins with given sign [8].

The phase transition in ordinary percolation (OP), where bonds or sites are added at random, represents a broad universality class. Recently Achlioptas *et al.* [9] made a simple modification by, at each step, selecting among two possibilities the link that leads to the slowest growth of large clusters. This global choice introduces a large length scale – the system size  $L$  – which can alter universality. Indeed, they concluded that an unusual, discontinuous transition (called “explosive percolation”) emerges where a macroscopic cluster appears suddenly while at the same time scaling in other quantities is observed [10–12]. Various modifications of the rule have been made [13, 14] – all finding evidence of a discontinuous transition. Although the claim for discontinuity in [9] was refuted later in [15], explosive percolation in that case does represent a new universality class.

Here we discuss a percolation process that also contains a potentially large length scale in its definition, in this case

the correlation length  $\xi$ . Our process has direct application to the study of complex networks. Instead of adding bonds randomly, we pick a random cluster and add bonds to its entire surface in order to link it to all adjacent clusters. Starting with the state where all clusters have size one, at each update  $t \rightarrow t + 1$  the process repeats until the entire lattice (or graph) is reduced to a single cluster. We call this “agglomerative percolation” (AP), in analogy with cluster growth by aggregation [16]. Thus if by chance a cluster of length scale  $\ell$  is picked, links are added simultaneously at distances  $\mathcal{O}(\ell)$  apart.

AP can be analyzed on any graph. It corresponds to random sequential renormalization [17] of a network, where a single cluster is identified as a ‘super’-node that is a local coarse-graining of the graph. In this perspective, scaling laws seen in renormalization studies of small-world networks [18–22] are a consequence of an AP phase transition and *do not indicate fractality of the underlying graph* [17, 23]. Previously AP was studied on critical trees [17] and in one dimension [23]. Scaling laws were found both analytically and numerically – but no phase transition occurs since both graphs have a topological dimension of one.

In order to establish the phase transition in AP, and its relationship to OP, we analyze it in two dimensions (which is clearly not a fractal graph), where many exact results for OP are known. We consider both square and triangular lattices. Clusters can be chosen with equal probability,

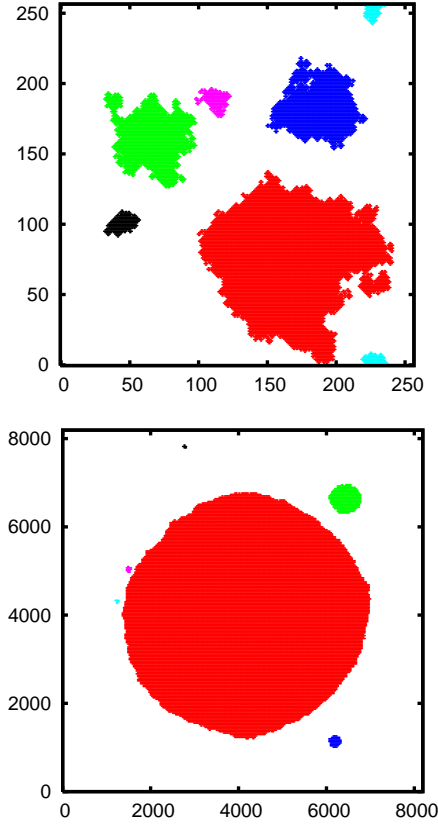


Fig. 1: (Color online) The six largest clusters in mass-weighted AP (model (b)) on a square lattice with  $L = 256$  (top), and  $L = 8192$  (bottom). The average cluster mass is two, or  $n = N/L^2 = 0.5$ . The red cluster is close to wrapping. All large clusters are compact.

or we can make biased choices according to the mass, radius, etc. of the clusters. Here both (a) uniform probabilities and (b) probabilities proportional to the cluster mass are studied. Model (b) coincides with choosing *sites* uniformly, and growing the whole cluster in which they lie. It shows runaway behavior resembling a first order transition with compact clusters – as in Ref. [24] (see Fig. 1). Model (a) is more subtle. Although clusters appear fractal (see Fig. 2) and the overall character resembles OP, fundamental differences arise. Most conspicuous is an unexpected difference between the two lattice types: While model (a) on the square lattice is definitely not in the OP universality class (e.g. the average cluster size diverges at the transition), the triangular lattice shares the same critical exponents as OP. We believe this violation of universality must be related to the long range nature of the growth process for large clusters.

We use  $L \times L$  lattices, with  $2^5 \leq L \leq 2^{14}$ . Boundary conditions are helical: periodic in the  $y$  direction while the right neighbor of site  $(x = L, y)$  is  $(x' = 1, y' \equiv y(\text{mod } L) + 1)$ . Diagonal bonds are added to obtain triangular lattices. We use an algorithm based on that in [25], augmented by a depth first search on the target cluster, in order to find all its neighboring clusters. The natural control parameter

in OP is  $p$ , the fraction of existing bonds or sites. In AP, however, the number of links is not uniquely defined, although it is a version of (correlated) bond percolation. One might join two clusters via a single link, but one might also put multiple links between them. Therefore, in AP it is more natural to use the average cluster number per site,

$$n = N/L^2 = \langle m \rangle^{-1} \quad , \quad (1)$$

where  $\langle m \rangle$  is the average cluster mass. For OP,  $n(p)$  is not analytic at  $p = p_c$ , but is monotonic with two continuous derivatives. Thus one can use  $n$  as the control parameter in OP and reproduce all known scaling laws. (We checked this explicitly; see also [10]). Instead of  $n$ , one might also use the number of agglomeration events,  $t$ , as the control parameter. In agreement with [17, 26], we found that  $t$  is more “noisy” than  $n$  and leads to slightly less clear results. We analyze the distribution  $P_n(m)$  of cluster masses  $m$  in configurations with cluster density  $n$ , and the probability  $p_{\text{wrap}}(n)$ , that a cluster wraps the torus in the  $y$ -direction, for each  $L$ .

In model (b), the growth rate for a cluster of mass  $m$  accelerates steeply with  $m$ , leading to a runaway effect. A cluster’s chance to be selected is  $m/L^2$ . Once chosen it grows all along its perimeter. Since most of its neighbors are small, it grows into a compact shape. For any  $0 < \alpha < 2$  and for  $L \rightarrow \infty$ , we conjecture that the largest cluster reaches mass  $m \sim L^\alpha$  at a time when  $\langle m \rangle \rightarrow 1$ . This “incipient” cluster continues to separate in mass from the others. It wraps the torus when  $m/L^2 = \mathcal{O}(1)$ . Thus an infinite incipient cluster appears at density  $n_c \rightarrow 1$  in the limit  $L \rightarrow \infty$ , while wrapping occurs much later, at  $0 < n_c < 1$ . Fig. 1 shows the six largest clusters in a typical run on a square lattice for both small and large  $L$  at  $n = 0.5$ . These snapshots were taken at a time long past the appearance of the incipient cluster and long before it wraps. One sees that the giant cluster becomes more dominant over all other clusters as  $L$  increases. Although convergence of  $n_c \rightarrow 1$  as  $L \rightarrow \infty$  is slow, it is in perfect agreement with numerical simulation results (data not shown). The same scenario holds for the triangular lattice.

In OP, cluster perimeters are for large clusters proportional to their mass. Thus, if a new bond is added at each time step, the average growth rate of a cluster is roughly  $dm/dt \propto m$ . In AP model (a) – where clusters are picked with uniform probability – those chosen grow by an amount proportional to their perimeter, so again (roughly)  $dm/dt \propto m$ . This leads neither to a runaway of large clusters as in model (b) nor to the retardation of their growth as in the Achlioptas process. Therefore, model (a) and OP cannot be distinguished by such a crude argument and their relationship could conceivably depend on microscopic details such as the type of lattice.

For model (a), we first consider square lattices. By eye individual configurations look like OP. However, wrapping thresholds depend strongly on  $L$ . Figure 2 displays the

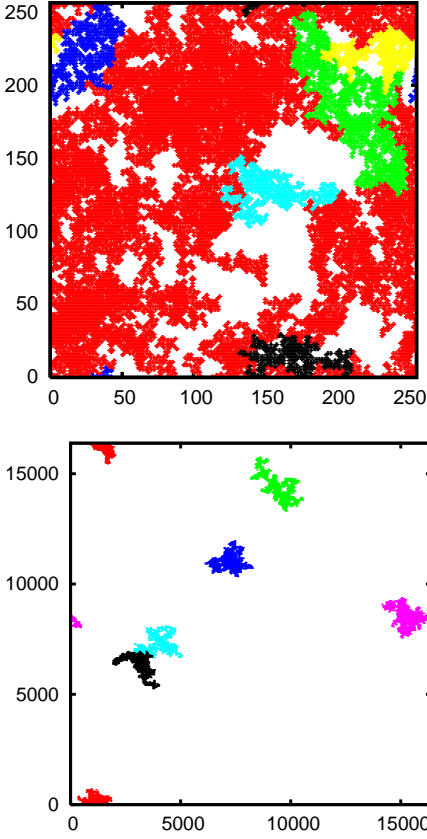


Fig. 2: (Color online) The six largest clusters in typical runs of AP model (a) on square lattices for  $n = N/L^2 = 0.1 = 1/\langle m \rangle$ . Top panel: for  $L = 256$ , the red cluster has already wrapped; Bottom panel: for  $L = 16384$ , all clusters are far from wrapping. In both cases, clusters appear fractal.

six largest clusters in a typical run when  $n = 0.1$  for both  $L = 256$  (top) and for  $L = 16384$  (bottom). While the largest cluster clearly wraps the small lattice, it is far from this point on the large one. Figure 3 shows the density  $n_{c,wrap}(L)$  at which half the runs contain a wrapping cluster. The data fall roughly on a straight line on a log-log plot. If deviations from a straight line were typical finite size corrections, this would mean that the average cluster size at the wrapping threshold diverges as a power of  $L$  when  $L \rightarrow \infty$ . However, for reasons explained below, we believe that  $n_{c,wrap}(L) \rightarrow 0$  logarithmically as  $L \rightarrow \infty$  (for explicit fits, see the supplementary material [27]). This implies that the correlation length exhibits an essential singularity as  $n \rightarrow n_c = 0$ .

Mass distributions  $P_n(m)$  for  $n = n_{c,wrap}(L)$  are displayed in Fig. 4 using a data collapse method which compares  $m^\tau P_n(m)$  to  $m/L^D$ , with  $\tau = 1.926$  and  $D = 1.96$ . A perfect collapse corresponds to a finite size scaling (FSS) *ansatz*

$$P_n(m) = m^{-\tau} f(\psi(n, L), m/L^D) \quad , \quad (2)$$

which generalizes the standard FSS *ansatz* [2] where  $\psi(n, L) = (n - n_c)L^{1/\nu}$ . Except for peak heights the data collapse is excellent. The apparent values for  $\tau$  and  $D$

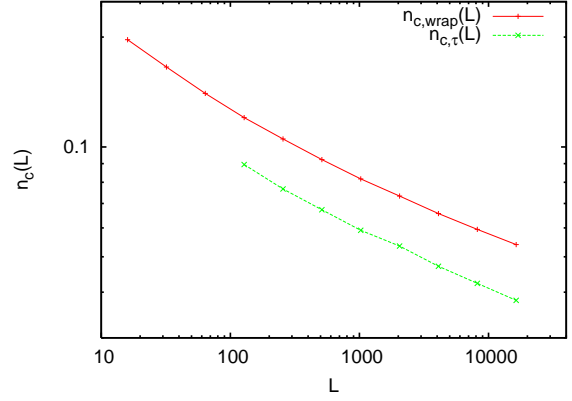


Fig. 3: (Color online) Critical densities  $n_c(L)$  vs.  $L$  for AP model (a) on square lattices. The upper curve is the average wrapping threshold. The lower one is the density at which the power law range in the cluster mass distribution extends furthest. Error bars are smaller than the symbol size.

deviate significantly from their values in OP ( $\tau = 2.055$  and  $D = 1.89$  in two dimensions). At  $n = n_{c,wrap}(L)$ ,  $m^\tau P_n(m)$  is not horizontal over a wide range of masses. Hence  $n = n_{c,wrap}(L)$  is not equal to  $n_{c,\tau}(L)$ . The latter is the density at which a power law in  $P_n(m)$  extends over the broadest range. Curves for  $m^\tau P_{n=n_{c,\tau}(L)}(m)$  are also shown in Fig. 4 and exhibit data collapse with a remarkably wide power law range. The power law regime describes the relatively few remaining clusters in configurations dominated by one wrapping cluster. The values  $n_{c,\tau}(L)$  are also shown in Fig. 3. and decrease similarly to  $n_{c,wrap}(L)$  as  $L$  increases.

If  $\tau \leq 2$  as suggested by Fig. 4, then the average cluster size at criticality diverges as  $L \rightarrow \infty$ , in agreement with Fig. 3 but in stark contrast to OP. Accepting this, the two possible (scaling) scenarios are  $\tau = 2$  or  $\tau < 2$ . If  $\tau < 2$ ,  $n_c(L)$  vanishes as  $\sim L^{-\delta}$  with some  $\delta > 0$ , and apparent values for  $\tau$  and  $D$  should not vary much with  $L$ . Neither of these statements is correct. Figure 3 shows definite curvature, and the best fit values for  $\tau$  and  $D$  both increase slightly but significantly with  $L$ , see Fig. 5 and [27]). While these small shifts are not visible on the scales shown in Fig. 4, they do not diminish as  $L$  increases. One would not expect to see large corrections to scaling that could explain Fig. 5 if  $\tau > 2$  since in that case the average cluster size is finite.

Since the numerical value of  $D$  is determined from the positions of the peaks in Fig. 4,  $D$  is actually the fractal dimension of the largest cluster. The contribution to  $\langle m \rangle$  from this cluster is  $s_{\max}/N = s_{\max}\langle m \rangle/L^2$ . If one assumes Eq. (2) and  $\tau < 2$ , then one can show that the largest cluster makes a non-vanishing contribution to  $\langle m \rangle$  as  $L \rightarrow \infty$ . This can only happen if  $s_{\max} \sim L^2$ , showing that the largest cluster has  $D = 2$  if  $\tau < 2$ . Furthermore, one would expect convergence of the apparent  $D$  to follow a power law in that case. But the slow convergence of  $D$  from below indicates again that the behavior is dominated by

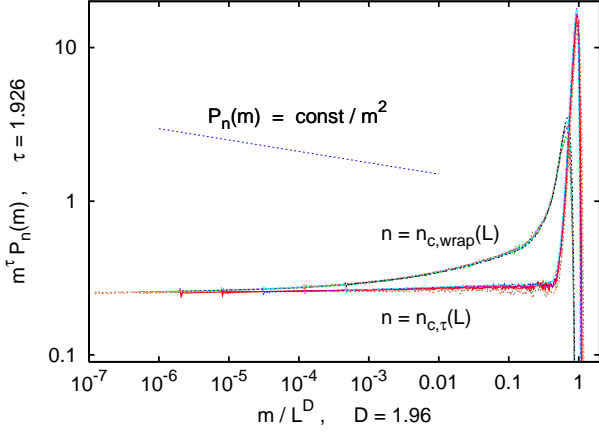


Fig. 4: (Color online) Data collapse for AP model (a) on square lattices:  $m^\tau P_n(m)$  vs.  $m/L^D$  with  $\tau = 1.926$  and  $D = 1.96$ . The curves with the smaller peaks are for  $n = n_{c,\text{wrap}}(L)$ , while the others are for  $n = n_{c,\tau}(L)$ . The first are at the wrapping threshold, while the second are when  $P_n(m)$  has the broadest power law range. The straight tilted line corresponds to  $P_n(m) \propto m^{-2}$ . System sizes are  $L = 256, 512, \dots, 16384$ .

logarithms.

We conclude thus that the true asymptotic values are  $\tau = D = 2$  and  $\nu = \infty$ . In addition, we measured the exponent  $\sigma$  [2]. In agreement with the scaling relation  $\sigma = D/\nu$ , we found  $\sigma \approx 0$  with rather slow convergence. Thus, all scaling relations are (trivially) satisfied (notice that the other exponents give no constraint, in the present case, on the order parameter exponent  $\beta$ ).

For the triangular lattice, clusters look like those in OP. Both  $n_{c,\tau}(L)$  and  $n_{c,\text{wrap}}(L)$  converge rapidly to the same (finite) critical value  $n_c = 0.1561 \pm 0.0002$ . Indeed, the best estimates of  $\tau(L)$ , obtained by fitting power laws to  $P_{n=n_{c,\tau}(L)}(m)$ , also converge rapidly to  $\tau = 2.057 \pm 0.002$ , in perfect agreement with OP (see Fig. 5). Also  $D$ , obtained from a data collapse as in Fig. 4, and the exponent  $\sigma$  are both within error equal to their values in OP, although these error bars are larger than for  $\tau$ , see [27].

Small apparent inconsistencies with OP arise when we try to estimate  $\nu$  using the scaling hypotheses  $n_{c,\tau}(L) - n_c \sim L^{-1/\nu}$  or  $p_{\text{wrap}}(n) = \phi[(n - n_c)L^{1/\nu}]$ . The first relation gives  $\nu = 1.10 \pm 0.07$ , significantly smaller than the value  $\nu = 4/3$  for OP. The second one gives  $\nu = 1.47 \pm 0.05$  for  $n \gtrsim n_c$ . Taken at face value, these estimates would exclude universality with OP. But we believe that they are artifacts of large finite size corrections. Figure 6 shows an attempted data collapse for  $dp_{\text{wrap}}(n)/dn$ . While the collapse is acceptable for  $n > n_c$ , huge tails develop for  $n \ll n_c$  as  $L$  increases. For small  $n$  these tails decay roughly as  $[(n_c - n)L^{0.6}]^{-\mu}$  with  $\mu \approx 1.5$ . We checked explicitly that the tails result from events where wrapping happened when a large target cluster was hit. In such cases,  $N$  can make huge jumps, so that the largest  $n$  at which the cluster wraps is far below the actual threshold. Thus we conclude that AP on the triangular lattice is in the OP universality class, with the caveat that the def-

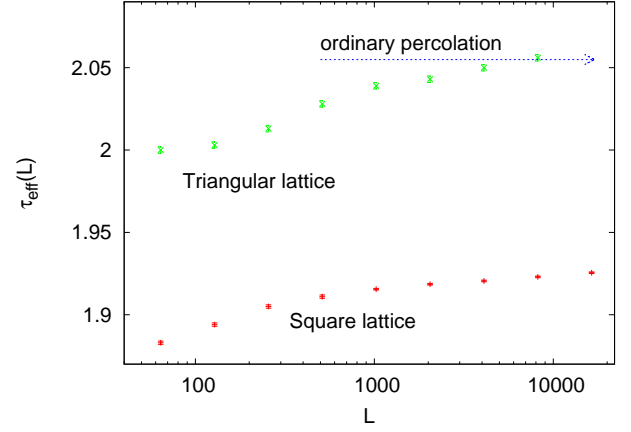


Fig. 5: (Color online) Plots of  $\tau_{\text{eff}}(L)$ , the effective Fisher exponents estimated from the longest stretches in  $P_n(m)$  that are compatible with pure power laws. As these estimates are somewhat subjective, the error bars are subjective as well. But their order of magnitude is consistent with the smoothness of the data with varying  $L$ . Clearly there is non-trivial  $L$  dependence both for the triangular and for the square lattice, with  $\tau_{\text{eff}}(L)$  increasing with  $L$  in both cases. While  $\tau_{\text{eff}}$  is compatible with the value  $\tau = 187/91 = 2.0549 \dots$  in the case of the triangular lattice, it is much smaller for the square lattice.

inition of  $n$  is affected by occasional large jumps which do not modify the main critical exponents but which do modify the tails of scaling functions as in Fig. 6.

For an appreciation how different the behaviors are on the square and triangular lattices, we show in Fig. 7 effective critical cluster densities. We plot them against  $L^{-3/4}$ , since this should give for OP straight lines, according to the FSS ansatz. While the data for the triangular lattice indeed follow roughly straight lines and give a finite non-zero value of  $n_c$ , the same is definitely not true for the square lattice: Those curves strongly bend down for  $L \rightarrow \infty$ , suggesting that  $n_c = 1/\nu = 0$ . Whether the latter is correct or not, this figure should leave no doubt that the square lattice model is not in the ordinary percolation universality class.

In summary, we have studied agglomerative percolation (AP) in two dimensions. This class of models is equivalent to random sequential renormalization schemes [17] first introduced to scrutinize the supposed fractality [18–22] of real world – in particular, small-world – networks. Regular lattices were chosen for two reasons: (1) They are not fractal; (2) Detailed comparison can be made with exact results for ordinary percolation (OP). Our results display some of the rich behavior possible in this general class of models and indicate that at least some of the scaling behavior found in [18–22] is due to AP rather than any supposed fractality of the underlying graph. If clusters are chosen with a bias for larger mass (model (b)), a runaway effect separates the largest cluster from the others and the behavior is completely different from OP. If clusters are chosen with equal probability (model (a)), then only



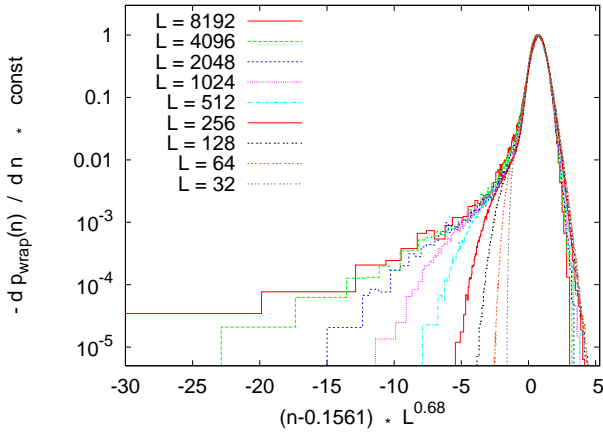


Fig. 6: (Color online) Data collapse plot for the probability density  $dp_{\text{wrap}}(n)/dn$  against  $(n - n_c)L^{1/\nu}$  with  $n_c = 0.1561$  and  $\nu = 1.47$  for model (a) on triangular lattices. The scale on the  $y$ -axis is adjusted such that all curves peak at  $y = 1$ .

a detailed numerical scaling analysis shows that AP is not in the OP universality class on the square lattice. On the other hand, AP on the triangular lattice shares critical exponents with OP.

AP may have applications beyond network renormalization. Growing clusters appear in many different physical situations. It could happen that further growth is triggered by some excitation where the entire cluster suddenly invades neighboring clusters at its boundary. Agglomerative percolation could also describe the growth of countries or urban areas. Countries often grow by overrunning and incorporating neighbors during aggressive periods, when they attack and incorporate simultaneously several of their neighbors.

Acknowledgement: We thank Bob Ziff for correspondence that helped us sharpen some of our arguments.

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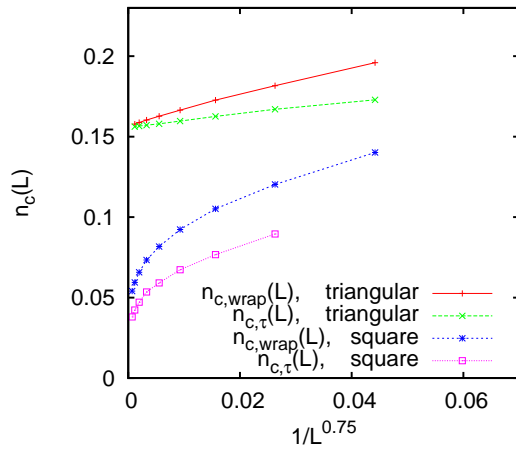


Fig. 7: (Color online) Effective critical cluster densities for model (a) versus  $L^{-3/4}$ , where  $L$  is the lattice size. For ordinary percolation, where  $n_c(L) - n_c \sim L^{-1/\nu}$  with  $\nu = 4/3$ , this should give straight lines. For each lattice type (triangular: upper pair of curves; square: lower pair of curves) we show results obtained with two different operational definitions for the critical point: (i) Maximal range of the power law  $P_n(m) \sim m^{-\tau}$ , and (ii) the probability to have a cluster that wraps around a lattice with helical boundary conditions is equal to  $1/2$ . The corresponding values of  $n_c(L)$  are called  $n_{c,\tau}(L)$  and  $n_{c,\text{wrap}}(L)$ . Error bars are typically of the size of the symbols.

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## Supplementary Material for “Agglomerative Percolation in Two Dimensions”

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(Dated: November 10, 2011)

In this supplementary material, we present several plots. They illustrate claims for which in the main paper either the actual data were not shown or they were plotted in different ways. The present plots are to convince the reader that:

- Model (b) (where clusters are selected for agglomeration according to their mass) develops runaway clusters very early in the process, leading to a “phase transition” at  $\langle m \rangle = 1$  in the infinite system limit  $N \rightarrow \infty$  (here,  $\langle m \rangle$  is the average mass per cluster) (Fig. S1);
- The same is not true for model (a), but model (a) shows qualitatively different behaviors on the square and triangular lattices (Figs. S2 to S5);
- In particular, while model (a) on the triangular lattice is in the ordinary percolation universality class, it is definitely not in that universality class on the square lattice;
- For the square lattice, *superficial analyses* would give a Fisher exponent  $\tau$  and a fractal dimension  $D$  that are both smaller than 2, but a more careful analysis, combined with analytic arguments, leads to the conclusion that  $\tau = D = 2$  exactly.

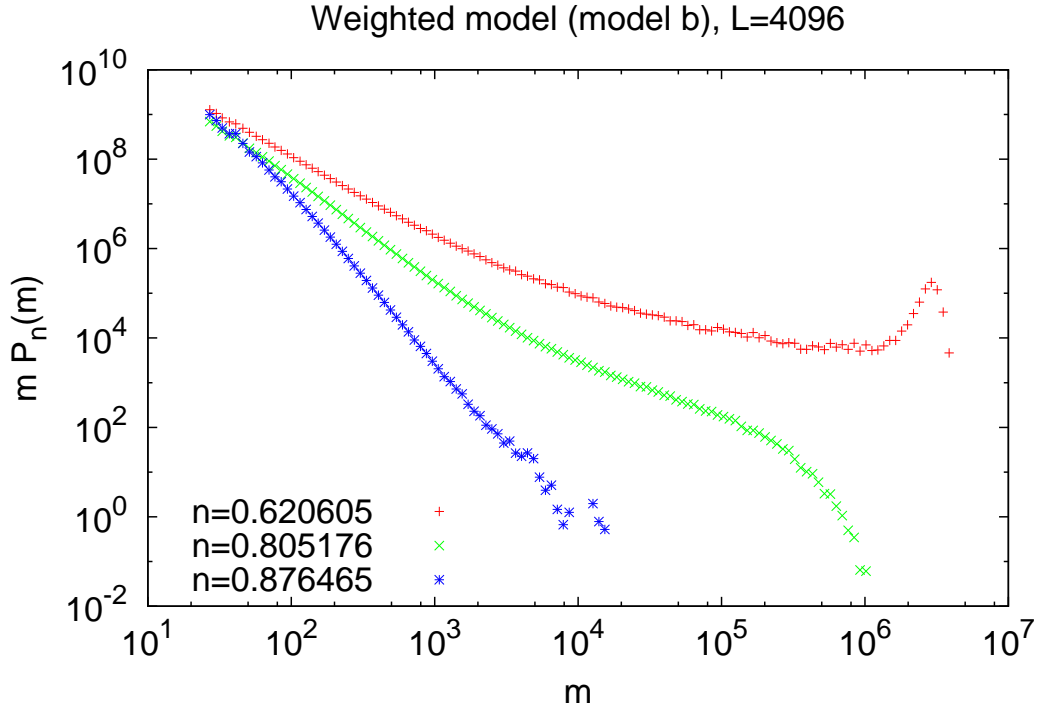


FIG. 1. (Color online) Mass distributions for model (b) with  $L = 4096$ . Each curve corresponds to a fixed value of  $n$  (number of clusters per site), corresponding to an average cluster mass  $\langle m \rangle = 1/n$ . The uppermost (red) curve shows a clear narrow peak, corresponding to a single cluster that involves a substantial (and thus not strongly fluctuating) fraction of all sites. The middle (green) curve shows a broad bump at its upper end, corresponding still to a runaway cluster in most realizations, albeit with a much wider mass range centered typically at a few percent of the total mass. The lowest (blue) curve corresponds to  $\langle m \rangle \approx 1.14$ , i.e. by far most clusters consist of a single site. In spite of this, it shows a long tail implying that *some* clusters have reached size  $10^4$  and beyond. In addition to Fig.1 in the main paper, this shows most directly the existence of runaway clusters at values of  $\langle m \rangle$  that converge to 1 as  $N \rightarrow \infty$ .

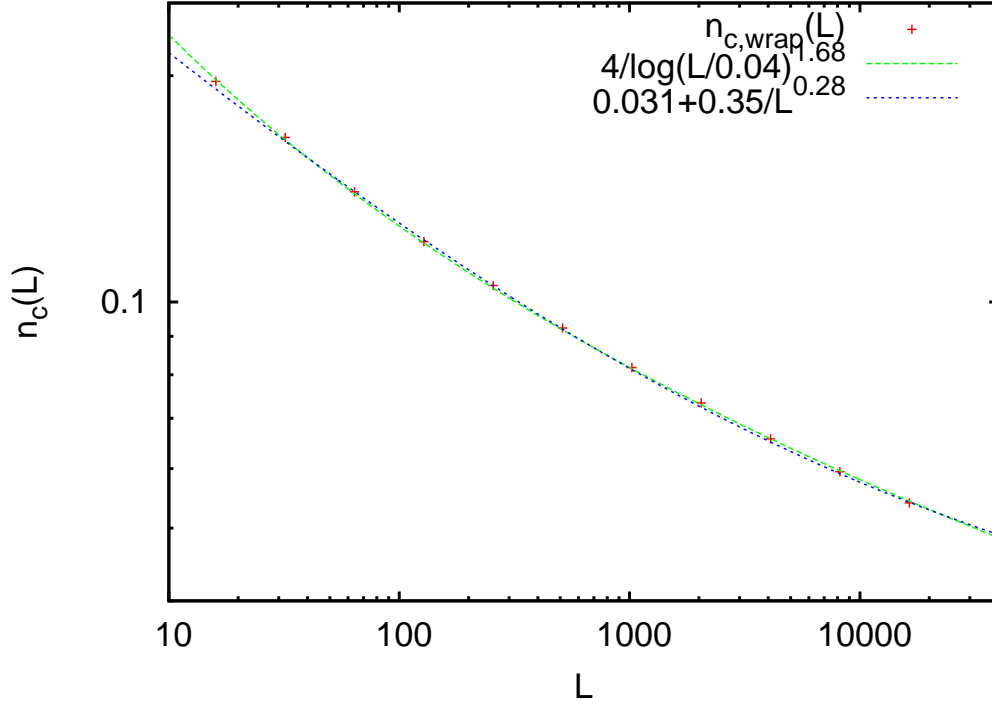


FIG. 2. (Color online) Plot of  $n_{c,\text{wrap}}(L)$  versus  $\log L$  for model (a) on the square lattice, together with two typical fits. One fit (green curve) is logarithmic,  $n_{c,\text{wrap}}(L) = 4/\ln(25L)^{1.68}$ . It assumes that  $n_{c,\text{wrap}} \equiv n_{c,\text{wrap}}(L = \infty) = 0$ . The other fit (blue) assumes that  $n_{c,\text{wrap}}$  is non-zero, and that  $n_{c,\text{wrap}}(L)$  has power law finite size corrections:  $n_{c,\text{wrap}}(L) = 0.031 + 0.35/L^{0.28}$ . Notice that both fits involve three free parameters. The logarithmic fit is slightly better, but it is not clear how significant this is, since one should expect an infinite series of further correction terms in any such fit. Notice also that  $n_{c,\text{wrap}}$  is very small (i.e., the average cluster mass at criticality is huge) in the second fit, in striking contrast to other (bond, site) percolation models on any 2-d lattice. In our opinion, this is a stronger argument against the second fit than the quality of the fit itself. Finally, no decent fit is possible with a pure power law  $n_{c,\text{wrap}}(L) = a/L^\alpha$ , expected if  $\tau < 2$  – although fits with two or more powers with different exponents would be possible.



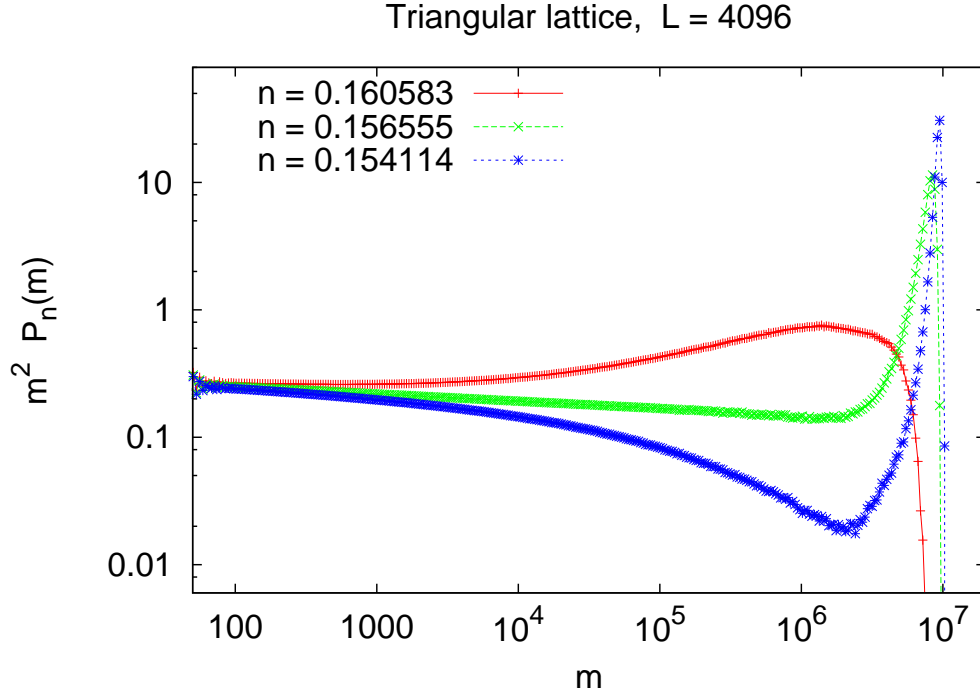


FIG. 3. (Color online) Mass distributions (log-log) for model (a) on the triangular lattice with  $L = 4096$ , for three different values of  $n$ . The middle (green) curve has the longest straight piece and is thus used to define  $n_{c,\tau}$ . The other two curves show the sub- (super-)critical behavior. For increased significance, we actually do not show  $P_n(m)$  but rather  $m^2 P_n(m)$ . The fact that the central (green) curve has a negative slope clearly indicates  $\tau > 2$ . Notice that this conclusion is rather robust and would not require a very precise estimate of  $n_{c,\tau}$ .

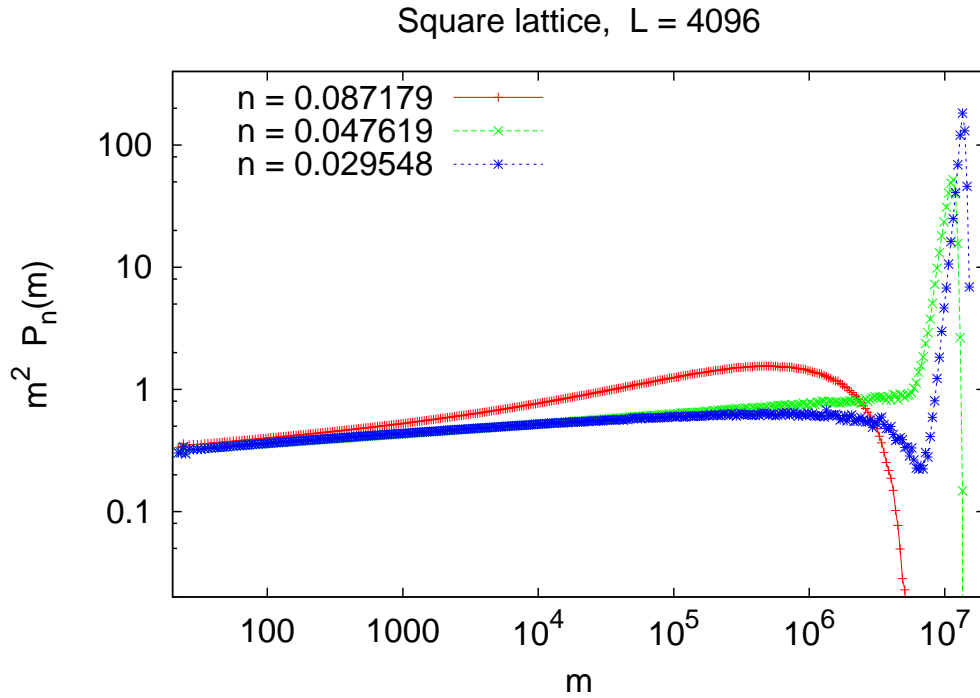


FIG. 4. (Color online) Same as Fig. S3, but for the square lattice. Superficially, the graph looks similar to Fig. S3, but the slope of the critical (green) curve now is positive, clearly suggesting that  $\tau < 2$  unless there are very strong corrections to scaling.

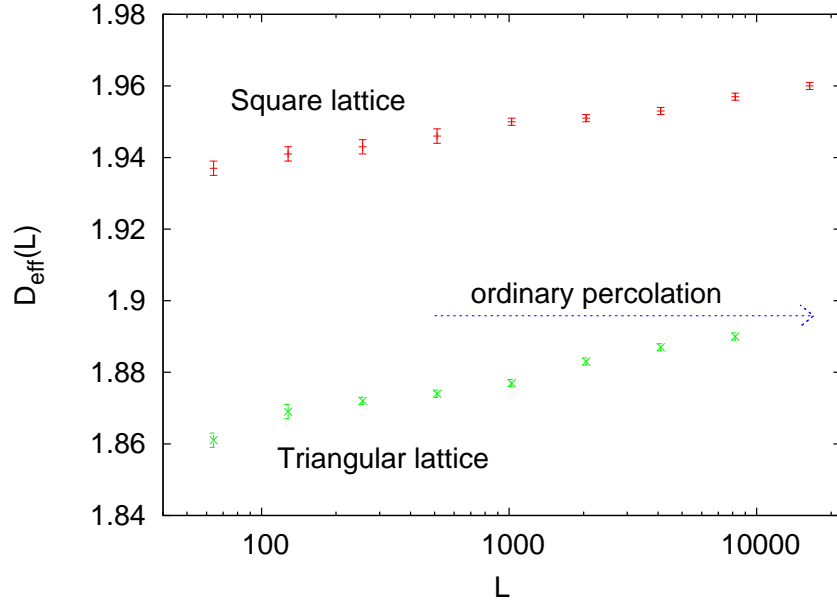


FIG. 5. (Color online) Similar to Fig. 5 of the main paper, but for the fractal dimension of the largest (percolating) cluster. It is obtained from the positions of the peaks of the central (critical) curves in mass distribution plots such as those in Figs. S4 and S5. Again we see that both curves increase with  $L$ . Again the triangular lattice data are compatible with ordinary percolation,  $D = 91/48 = 1.8958\dots$ , but the square lattice data are not. There, the percolating cluster has a distinctly larger dimension which is moving *away* from the ordinary percolation value. The analytical arguments given in the main paper allow only three possible scenarios: (i)  $D < 2, \tau > 2, \nu < \infty$ , and power law corrections to scaling; (ii)  $D = \tau = 2, \nu = \infty$ , and logarithmic corrections; and (iii)  $D = 2, \tau < 2, \nu < \infty$ , and power law corrections to scaling. Scenario (i) (i.e.,  $\tau > 2$ ) seems definitely ruled out for the square lattice by the data. Scenarios (ii) and (iii) are both possible, but would require very large and slowly vanishing finite size corrections for  $D_{\text{eff}}(L)$ . At face value, the data for  $n_c(L)$  shown in Fig. S2 and in Fig. 3 of the main paper would favor scenario (ii), while  $\tau_{\text{eff}}(L)$  (Fig. 5) would favor scenario (iii). Our final preference of scenario (ii) is based on the fact that it is the only one which can naturally accommodate logarithmic (and thus very slowly vanishing) finite size corrections.